Introduction to Graphical Models

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Reference

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- Jonathan S. Yedidia, William T. Freeman, and Yair Weiss, Understanding Belief Propagation and its Generalizations, 2001.

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- Jonathan S. Yedidia, Message-passing Algorithms for Inference and Optimization: “Belief Propagation” and “Divide and Concur”

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Inference problems and Belief Propagation

- Inference problems arise in statistical physics, computer vision, error-correcting coding theory, and AI.
- BP is an efficient way to solve inference problems based on passing local messages.

Bayesian networks

- Probably the most popular type of graphical model
- Used in many application domains: medical diagnosis, map learning, language understanding, heuristics search, etc.

Probability (Reminder)

Source: Wikipedia.org

- Sample space is the set of all possible outcomes.

  Example: S = {1,2,3,4,5,6}

- Power set of the sample space is obtained by considering all different collections of outcomes.

  Example Power set = {{},{1},{2},…,{1,2},…,{1,2,3,4,5,6}}

- An event is an element of Power set.

  Example E = {1,2,3}

Probability (Reminder)

- Assigns every event E a number in [0,1] in the following manner:

$$p(A) = \frac{|A|}{|S|}$$

- For example, let A = {2,4,6} denote the event of getting an even number while rolling a dice once:

$$p(A) = \frac{|\{2,4,6\}|}{|\{1,2,3,4,5,6\}|} = \frac{3}{6} = \frac{1}{2}$$

Conditional Probability (Reminder)

- If A is the event of interest and we know that the event B has already occurred then the conditional probability of A given B:

$$p(A|B) = \frac{p(A \cap B)}{p(B)}$$

- The basic idea is that the outcomes are restricted to only B then this serves as the new sample space.
- Two events A and B are statistically independent if

$$p(A \cap B) = p(A)p(B)$$

- Two events A and B are mutually independent if

$$p(A \cap B) = 0$$

Bayes Theorem (Reminder)

- Let A and B be two events and $p(B) \neq 0$.

$$p(A|B) = \frac{p(A)p(B|A)}{p(B)}$$

Reminder

Summary of probabilities

Event	Probability
A	$P(A) \in [0,1]$
not A	$P(A^{\complement}) = 1 - P(A)$
A or B	$P(A \cup B) = P(A) + P(B) - P(A \cap B)$ $P(A \cup B) = P(A) + P(B) \qquad \text{if A and B are mutually exclusive}$
A and B	$P(A \cap B) = P(A \vert B)P(B) = P(B \vert A)P(A)$ $P(A \cap B) = P(A)P(B) \qquad \text{if A and B are independent}$
A given B	$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{P(B \vert A)P(A)}{P(B)}$

Source: Wikipedia.org

A murder mystery

A fiendish murder has been committed

Whodunit?

There are two suspects:

- the **Butler**
- the **Cook**

There are three possible murder weapons:

- a butcher's **Knife**
- a **Pistol**
- a fireplace **Poker**

Slide Source: Christopher M. Bishop

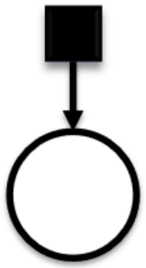
Prior distribution

Butler has served family well for many years
Cook hired recently, rumours of dodgy history

P(Culprit = **Butler**) = 20%

P(Culprit = **Cook**) = 80%

Probabilities add to 100%



P(Culprit)

Culprit = {**Butler**, **Cook**}

This is called a *factor graph*
(we’ll see why later)

Slide Source: Christopher M. Bishop

Conditional distribution

Butler is ex-army, keeps a gun in a locked drawer

Cook has access to lots of knives

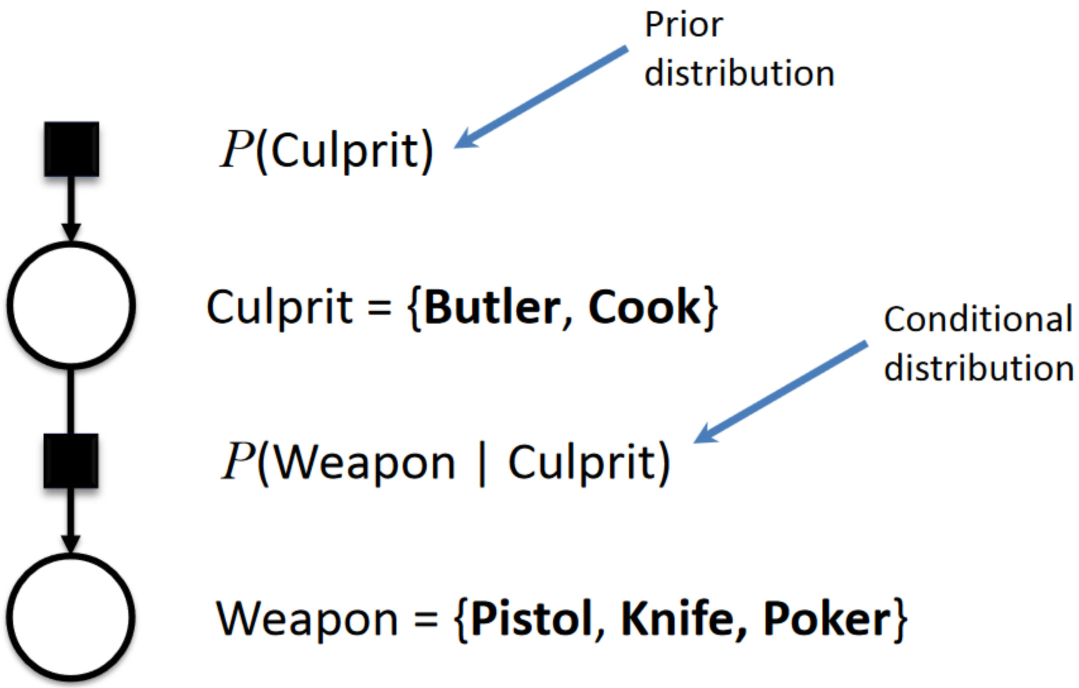
Butler is older and getting frail

	Pistol	Knife	Poker	
Cook	5%	65%	30%	**= 100%**
Butler	80%	10%	10%	**= 100%**

P(Weapon | Culprit)

Slide Source: Christopher M. Bishop

Factor graph



Slide Source: Christopher M. Bishop

Joint distribution

What is the probability that the **Cook** committed the murder using the **Pistol**?

P(Culprit = **Cook**) = 80%

P(Weapon = **Pistol** | Culprit = **Cook**) = 5%

P(Weapon = **Pistol** , Culprit = **Cook**) = 80% x 5% = 4%

Likewise for the other five combinations of Culprit and Weapon

Slide Source: Christopher M. Bishop

Joint distribution

	Pistol	Knife	Poker
Cook	4%	52%	24%
Butler	16%	2%	2%

= 100%

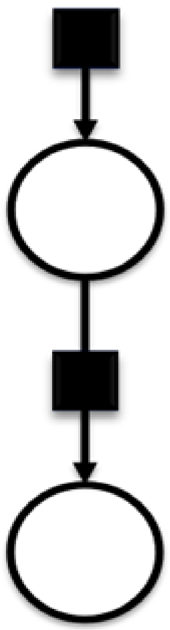
P(Weapon, Culprit) = P(Weapon | Culprit) P(Culprit)

$$\boxed{P(x, y) = P(y|x)P(x)}$$

Product rule

Slide Source: Christopher M. Bishop

Factor graphs



P(Culprit)

Culprit = {**Butler, Cook**}

P(Weapon | Culprit)

Weapon = {**Pistol, Knife, Poker**}

P(Weapon, Culprit) = P(Weapon | Culprit) P(Culprit)

Slide Source: Christopher M. Bishop

Marginal distribution of Weapon

	Pistol	Knife	Poker
Cook	4%	52%	24%
Butler	16%	2%	2%
	= 20%	**= 54%**	**= 26%**

$$\boxed{P(x) = \sum_y P(x, y)}$$

Sum rule

Posterior distribution

We discover a **Pistol** at the scene of the crime

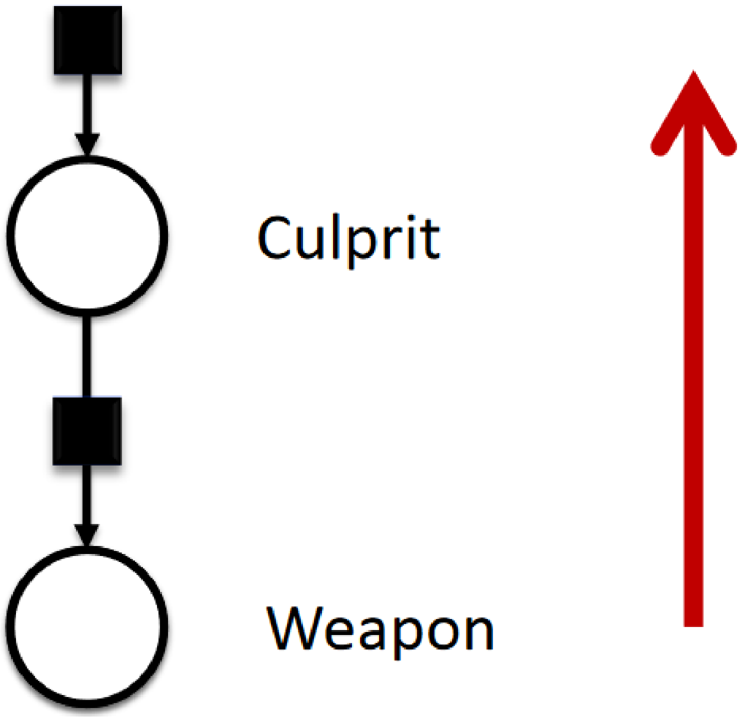
	Pistol	**Knife**	**Poker**	
Cook	4%	~~52%~~	~~24%~~	= 20%
Butler	16%	~~2%~~	~~2%~~	= 80%

This looks bad for the Butler!

P(Culprit = Cook | Weapon = Pistol) =
P(Culprit = Cook, Weapon = Pistol)/P(Weapon = Pistol) = 0.04/0.20 = 0.20

P(Culprit = Butler | Weapon = Pistol) =
P(Culprit = Butler, Weapon = Pistol)/P(Weapon = Pistol) = 0.16/0.20 = 0.80

Reasoning backwards



Bayes' theorem

$$P(x, y) = P(y|x)P(x)$$

likelihood

prior

$$P(y|x) = \frac{P(x|y)P(y)}{P(x)}$$

posterior

Prior – belief before making a particular obs.

Posterior – belief after making the obs.

Posterior is the prior for the next observation

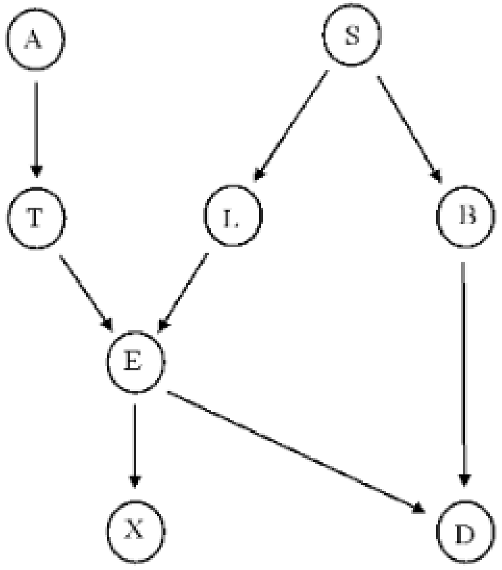
– Intrinsically incremental

Medical diagnosis problem

- We will have (possibly incomplete) information such as symptoms and test results.
- We would like the probability that a given disease or a set of diseases is causing the symptoms.

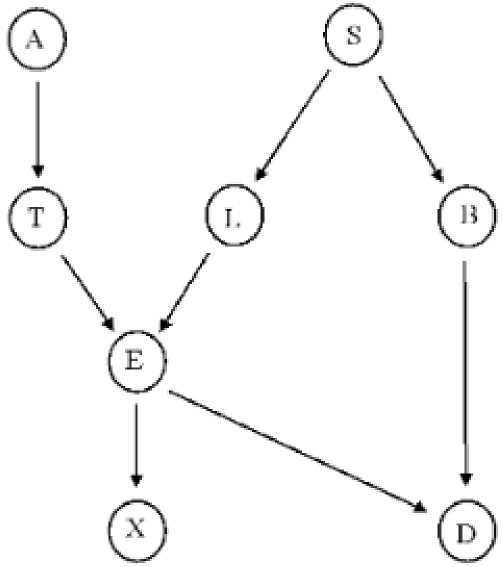
Fictional Asia example (Lauritzen and Spiegelhalter 1988)

- A recent trip to Asia (A) increases the chance of Tuberculosis (T).
- Smoking is a risk factor for both lung cancer (L) and Bronchitis (B).
- The presence of either (E) tuberculosis or lung cancer can be treated by an X-ray result (X), but the X-ray alone cannot distinguish between them.
- Dyspnea (D) (shortness of breath) may be caused by bronchitis (B), or either (E) tuberculosis or lung cancer.



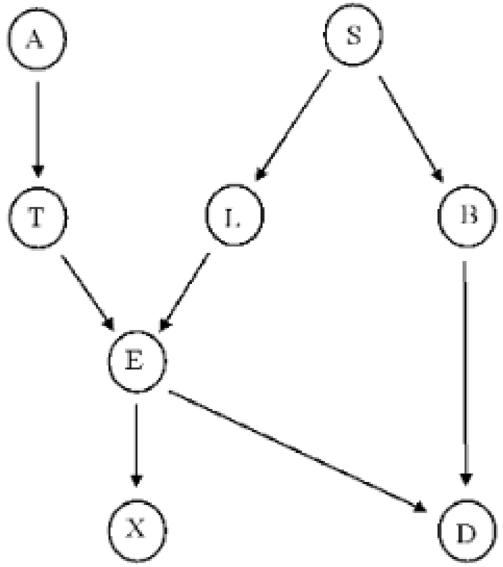
Each node represents a random variable
Arrows indicate cause-effect relationship

Bayesian networks



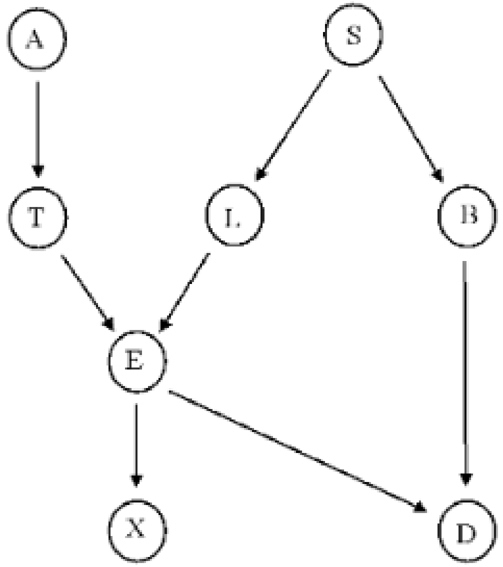
- Let x_i denote the different possible states of the node i.
- Associated with each arrow, there is a conditional probability.
- $p(x_L|x_S)$ denote the conditional probability that a patient has lung cancer given he does or does not smoke.

Bayesian networks



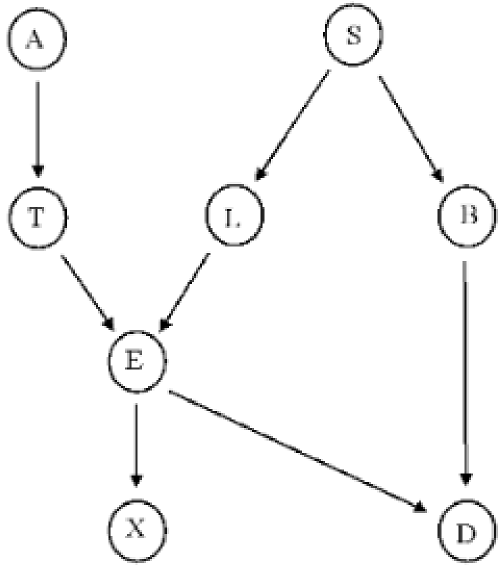
- $p(x_L|x_S)$ denote the conditional probability that a patient has lung cancer given he does or does not smoke.
- Here we say that "S" node is the parent of the "L" node.

Bayesian networks



- Some nodes like D might have more than one parent.
- We can write the conditional probability as follows
$$p(x_D|x_E, x_B)$$
- Bayesian networks and other graphical models are most useful if the graph structure is sparse.

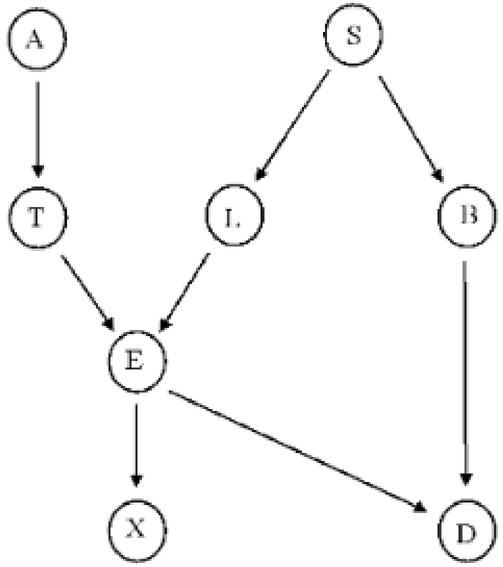
Joint probability in Bayesian networks



- The joint probability that the patient has some combination of the symptoms, test results, and diseases is given below:

$$p(\{\boldsymbol{x}\}) = p(\{x_A, x_S, x_T, x_L, x_B, x_E, x_X, x_D\})$$

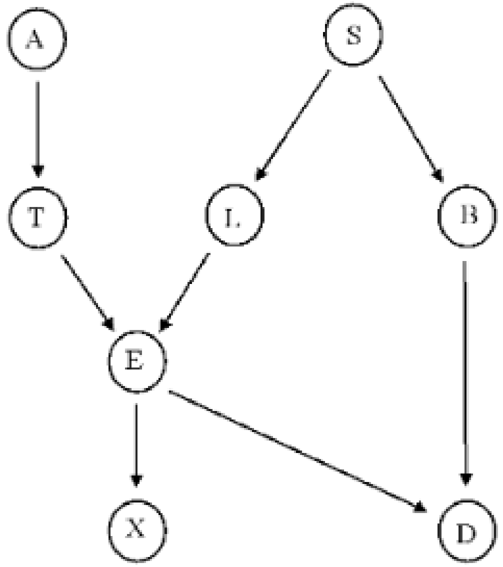
Joint probability in Bayesian networks



$$p(\{\boldsymbol{x}\}) = p(\{x_A, x_S, x_T, x_L, x_B, x_E, x_X, x_D\})$$

$$= p(x_A)p(x_S)p(x_T|x_A)p(x_L|x_S)p(x_B|x_S)p(x_E|x_T, x_L)p(x_X|x_E)p(x_D|x_E, x_B)$$

Joint probability in Bayesian networks



In general, Bayesian network is an acyclic directed graph with N random variables x_i that defines a joint probability function:

$$p(x_1, x_2, x_3, \dots, x_N) = \Pi_{\{i=1\}}^{\{N\}} p(x_i | Par(x_i))$$

Marginal Probabilities

- Probability that a patient has a certain disease:

$$p(x_N) = \sum_{x_1} \sum_{x_2} \dots \sum_{x_{\{N-1\}}} p(x_1, x_2, \dots, x_N)$$

- Marginal probabilities are defined in terms of sums of all possible states of all other nodes.
- We refer to approximate marginal probabilities computed at a node x_i as beliefs and denote it as follows:

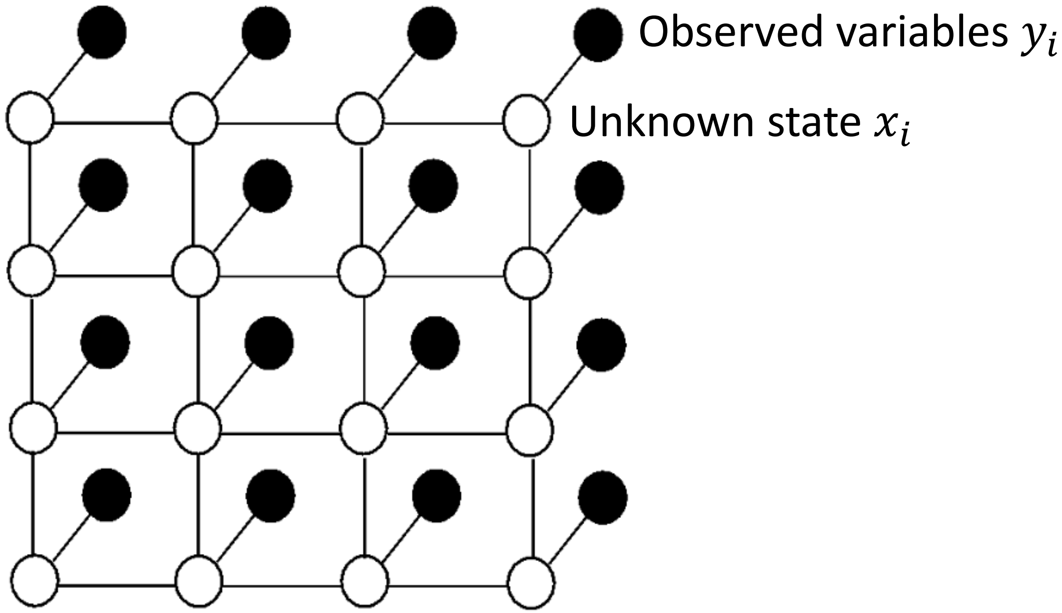
$$b(x_i)$$

- The virtue of BP is that it can compute the beliefs (at least approximately) in graphs that can have a large number of nodes efficiently.

Pairwise Markov Random Fields

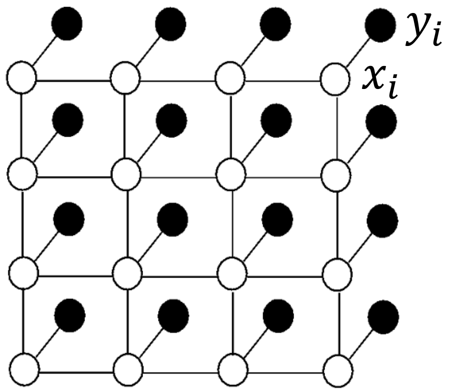
- Attractive theoretical model for many computer vision tasks (Geman 1984).
- Many computer vision problems such as segmentation, recognition, stereo reconstruction are solved.

Pairwise Markov Random Fields



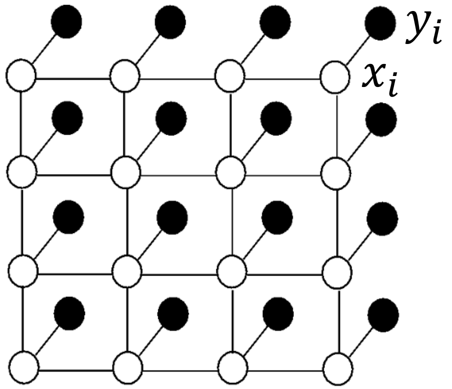
- In a simple depth estimation problem on an image of size 1000 x 1000, every node can have states from 1 to D denoting different distances from the camera center.

Pairwise Markov Random Fields



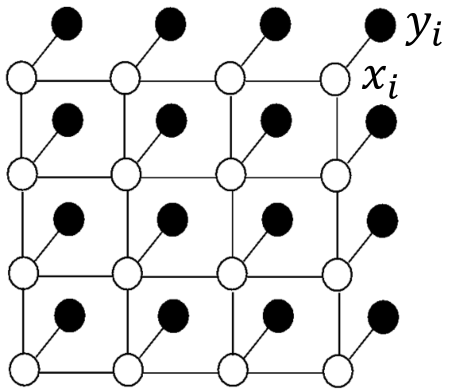
- Let us observe certain quantities about the image y_i and we are interested in computing other entities about the underlying scene x_i.
- The indices i denote certain pixel locations.
- Assume that there is some statistical dependency between x_i and y_i and let us denote it by some compatibility function $\phi_i(x_i, y_i)$, also referred to as the evidence.

Pairwise Markov Random Fields



- To be able to infer anything about the scene, there should be some kind of structure on x_i.
- In a 2D grid, x_i should be compatible with nearby scene elements x_j.
- Let us consider a compatibility function $\psi_{ij}(x_i, x_j)$ where the function connects only nearby pixel elements.

Pairwise Markov Random Fields



$$p(\{\boldsymbol{x}\},\{\boldsymbol{y}\}) = \frac{1}{\mathrm{Z}} \Pi_{\{ij\}} \psi_{ij}\left(x_i, x_j\right) \Pi_i \phi_i(x_i, y_i)$$

- Here Z is the normalization constant.
- The Markov Random fields is pairwise because the compatibility function depends only on pairs of adjacent pixels.
- There is no parent-child relationship in MRFs and we don't have directional dependencies.

Potts Model

- Potts model comes from statistical mechanics, where the Potts model consists of *spins* that are placed on a lattice. Each spin can take several discrete states, and there is interaction between nearby spins.
- In the MRF, the interaction $J_{ij}(x_i, x_j)$ between two neighboring nodes is given by

$$J_{ij}(x_i, x_j) = \ln \psi_{ij}(x_i, x_j)$$

- The field $h_i(x_i)$ at each node is given by

$$h_i(x_i) = \ln \phi_i(x_i, y_i)$$

Potts Model

- The Potts model energy is defined as below:

$$E(\{x_i\}) = -\sum_{ij} J_{ij}\left(x_i, x_j\right) - \sum_{i} h(x_i)$$

Boltzmann's law from statistical mechanics

- The pairwise MRF exactly corresponds to the Potts model energy at temperature T = 1.

$$p(\{x_i\}) = \frac{1}{Z} e^{-\frac{E(\{x_i\})}{T}}$$

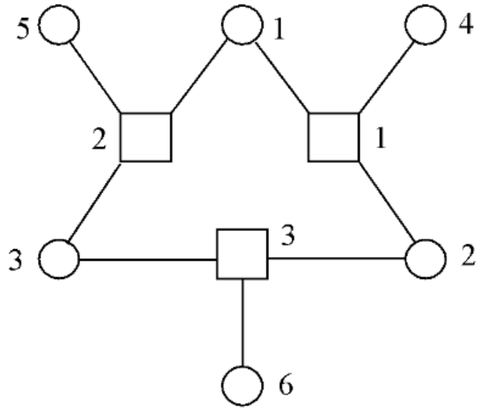
- The normalization constant Z is called the partition function.

ISING model

- If the number of states is just 2 then the model is called an ising model.
- The problem of computing beliefs can be seen as computing local magnetizations in Ising model.
- The spin glass energy function is written below using two-state spin variables $s_i = \{+1, -1\}$:

$$E(\{s_i\}) = -\sum_{ij} J_{ij}\left(s_i, s_j\right) - \sum_{i} h(s_i)$$

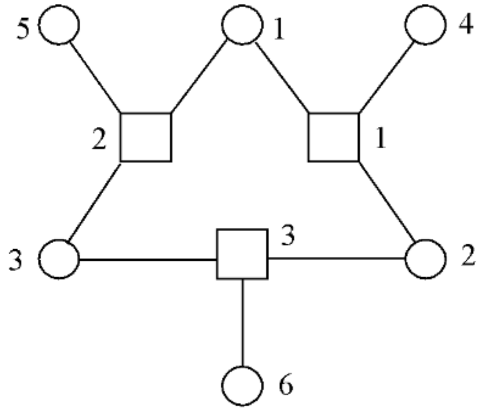
Tanner Graphs and Factor Graphs



We have transmitted N = 6 bits with k = 3 parity check constraints.

- Error-correcting codes: We try to decode the information transmitted through noisy channel.
- The first parity check code forces the sum of bits from #1, #2, and #4 to be even.

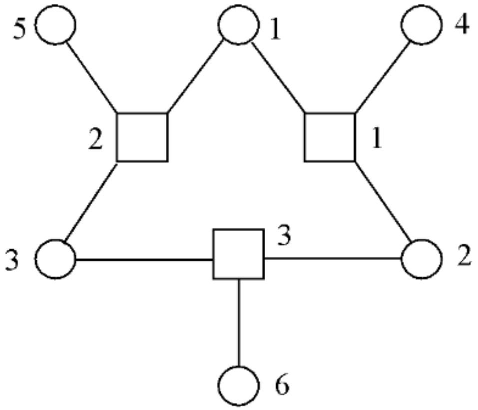
Tanner Graphs and Factor Graphs



We have transmitted N = 6 bits with k = 3 parity check constraints.

- Let $y_i$be the received bit and the transmitted bit be given by x_i.
- Joint probability can be written as follows:
- $p(\{x, y\}) = \frac{1}{Z}\psi_{124}(x_1, x_2, x_4)\psi_{135}(x_1, x_3, x_5)\psi_{236}(x_2, x_3, x_6)\Pi_i p(y_i|x_i)$

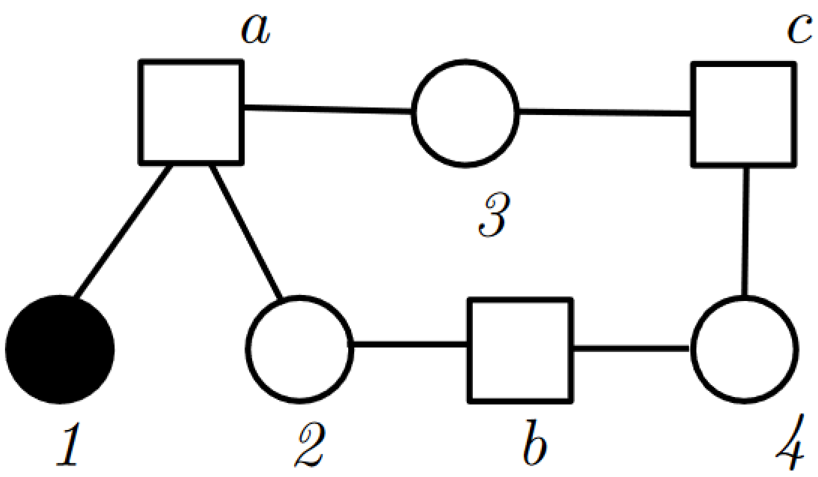
Tanner Graphs and Factor Graphs



We have transmitted N = 6 bits with k = 3 parity check constraints.

- The parity check functions have values 1 when the bits satisfy the constraint and 0 if they don't.
- A decoding algorithm typically tries to minimize the number of bits that are decoded incorrectly.

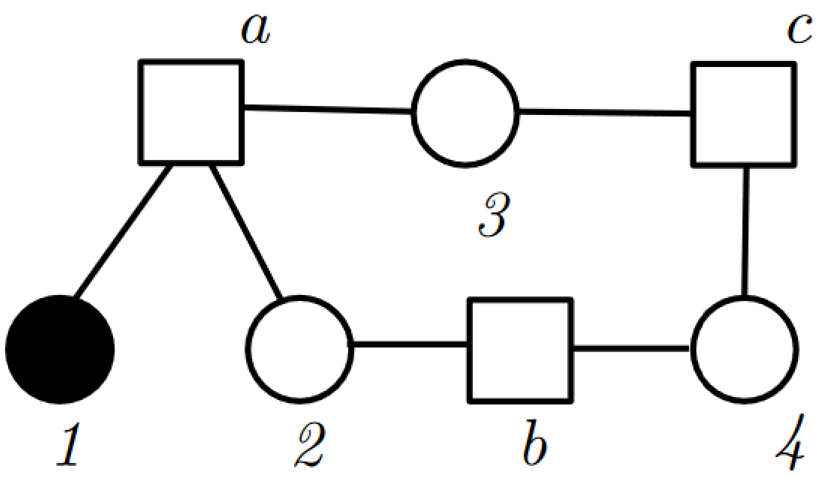
Factor Graphs (Using Energy or Cost functions)



Toy factor graph with one observed variable, 3 hidden variables, and 3 factor nodes

- Factor graphs are bipartite graphs containing two types of nodes: variable nodes (circles) and factor nodes (squares).

Factor Graphs (Using Energy or Cost functions)

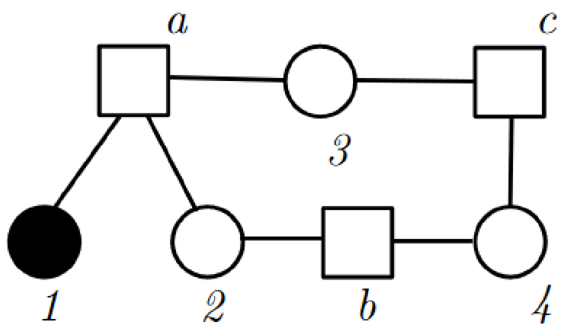


Toy factor graph with one observed variable, 3 hidden variables, and 3 factor nodes

- $C(x_1, x_2, x_3, x_4) = C_a(x_1, x_2, x_3) + C_b(x_2, x_4) + C_c(x_3, x_4)$

Factor Graphs (Using Energy or Cost functions)

x_1	x_2	x_3	C_a
0	0	0	∞
0	0	1	0
0	1	0	0
0	1	1	∞
1	0	0	0
1	0	1	∞
1	1	0	∞
1	1	1	0



x_2	x_4	C_b
0	0	1.2
0	1	1.7
0	2	3.2
1	0	1.9
1	1	0.6
1	2	1.4

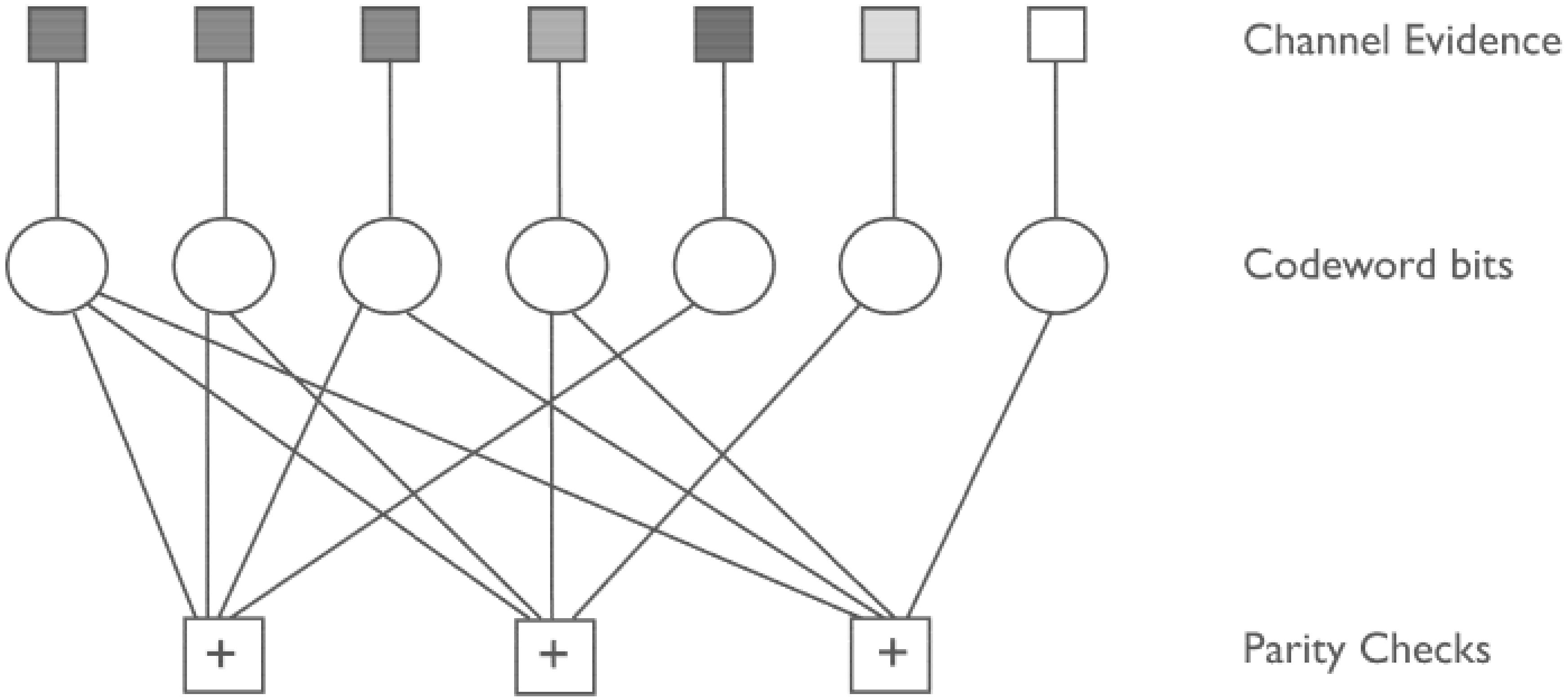
x_3	x_4	C_c
0	0	0.4
0	1	1.9
0	2	0.2
1	0	4.9
1	1	0.3
1	2	2.4

Lowest Energy Configurations

- $C(x_1, x_2, x_3, x_4) = C_a(x_1, x_2, x_3) + C_b(x_2, x_4) + C_c(x_3, x_4)$

- Finding the lowest energy state and computing the corresponding variable assignments is a hard problem
- In most general cases, the problem is NP-hard.

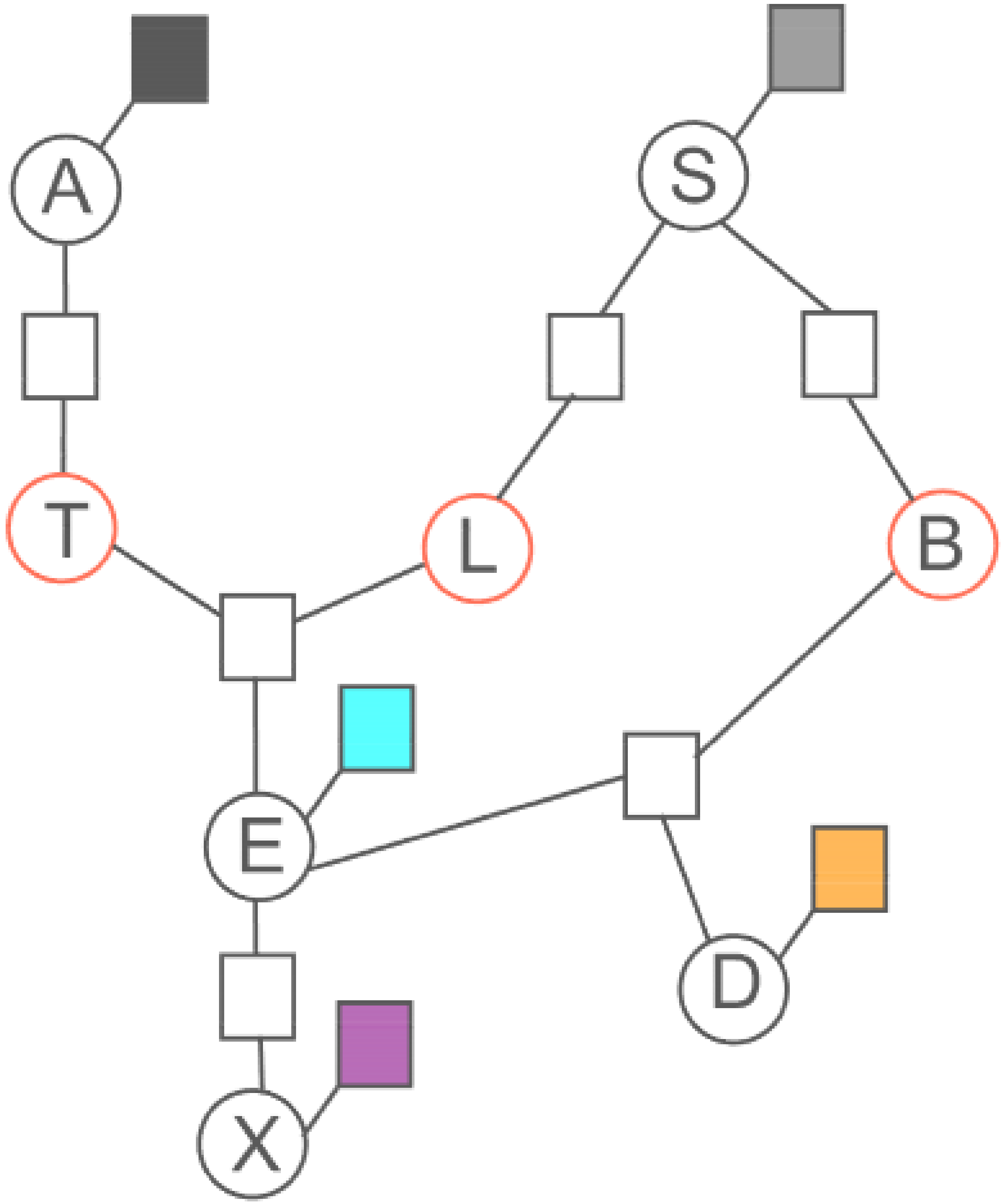
Factor Graphs for Error Correction



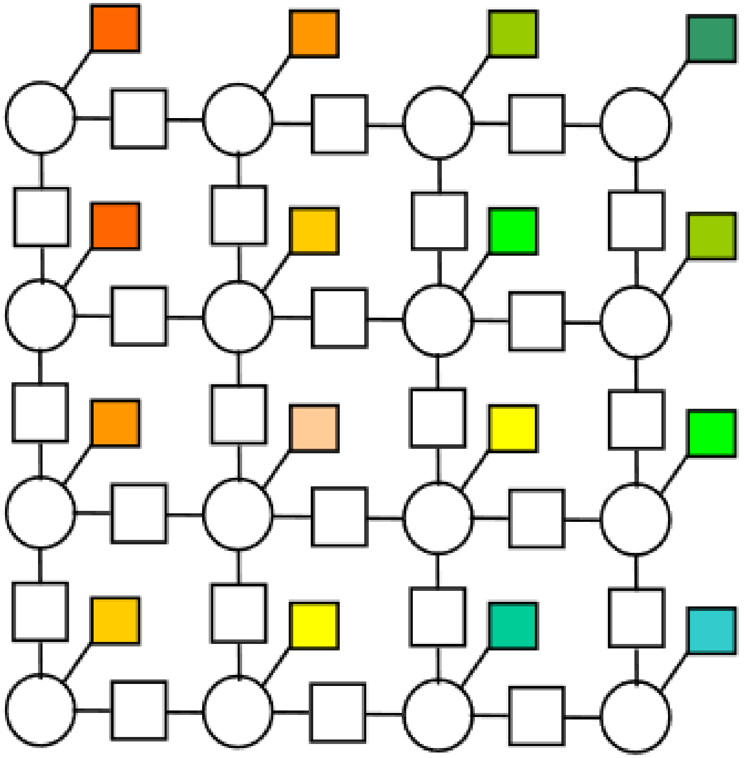
A factor graph for (N=7,k=3) Hamming code, which has 7 codeword bits, of the left-most four are information bits and the last 3 are parity bits.

Factor graph for the medical expert system

- Here the variables are given by Asia (A) , Tuberculosis (T), Lung cancer (L), Smoker (S), Bronchitis (B), Either (E), X-ray (X), and D.



Stereo reconstruction in Computer Vision



Set up the Factor graphs

- Point matching between 2 images given the Fundamental matrix.
- Point correspondences between 2 sets of 3D points.