Motion Estimation

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Presentation Outline

1. Review
2. Epipolar constraint
3. Fundamental Matrix
How do you get keypoint correspondence?

We use keypoint and descriptor matching algorithms, e.g., SIFT, BRIEF, etc.

What kind of constraints exist on the point correspondences in two images?

Epipolar constraint
How do you get keypoint correspondence?

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- We use keypoint and descriptor matching algorithms, e.g., SIFT, BRIEF, etc.
- What kind of constraints exist on the point correspondences in two images?
  - Epipolar constraint
1. Review

2. Epipolar constraint

3. Fundamental Matrix
What can you say about matching pixels?

- Assume that we are given the calibration, rotation, and translation parameters for the two cameras.
- We are given a single pixel $q_1$ in the left image.
- Let $q_2$ be the unknown pixel in the second image corresponding to $q_1$.
- Given $q_1$ can we find the location of $q_2$?
  - NO!
What can you say about matching pixels?

- For simplicity, we don’t show the optical axis.
What can you say about matching pixels?

- We consider different 3D points $Q^m$ on the backprojection of $q_1$.
- We look at the forward projections of these 3D points on the right image.
- The different projections are the different possibilities for $q_2$ given the position of $q_1$. 
What can you say about matching pixels?

- What is the parametric curve that passes through different possible locations of \( q_2 \)?
What can you say about matching pixels?

- It is a straight line.
What can you say about matching pixels?

What can you say if \( q_2 \) is given and we are interested in finding the location of \( q_1 \).
What can you say about matching pixels?

- Yes, it is also a straight line.
- Given a pixel in one image, the corresponding pixel in the other image is constrained to lie on a straight line.
Epipolar Plane and Epipoles

- **Epipolar plane** is the plane formed by the two camera centers \((O_1, O_2)\) and a 3D point \(Q^m\).

- The line joining the two camera centers intersect the image planes at points that we refer to as **epipoles**.

- The epipole in the first image is denoted by \(e_1\). The epipole in the second image is denoted by \(e_2\).
Given a pixel $q_1$, the corresponding pixel $q_2$ lies on a line in the right image that we refer to as epipolar line $l_2$. Note that this line passes through the epipole $e_2$.

The epipolar line in the first image is denoted by $l_1$ and it joins $q_1$ and $e_1$.

Note that the epipoles depend only on rotation, translation, and calibration parameters of the two cameras.
For every pair of matching pixels, we can think of an epipolar plane formed by the optical centers and the 3D point.

All the epipolar planes pass through the epipoles. Thus the epipolar lines can be seen as family of lines passing through a single point.
Derivation of the epipolar line

Given a pixel $q_1$, the corresponding pixel $q_2$ lies on epipolar line $l_2$.

The epipolar line $l_2$ in the right image is the line joining the $e_2$ and $q_2$ on the right image.

Let the forward projections be given by:

$q_1 \sim K_1 R_1 (l \mid -t_1) Q^m$. $q_2 \sim K_2 R_2 (l \mid -t_2) Q^m$. 
Derivation of the epipolar line

- The epipole $e_2$ is the projection of the left camera center on the right image. The left camera center is given by $t_1$.
- A 3D point on the back-projected ray of $q_1$ is given by $\lambda_1 R_1^T K_1^{-1} q_1 + t_1$. We obtain $q_2$ by projecting this point on the right image.
Derivation of the epipolar line

\[ e_2 \sim K_2 R_2 (l \ | \ - t_2) \begin{pmatrix} t_1 \\ 1 \end{pmatrix} \]

\[ q_2 \sim K_2 R_2 (l \ | \ - t_2) \begin{pmatrix} \lambda_1 R_1^T K_1^{-1} q_1 + t_1 \\ 1 \end{pmatrix} \]
Derivation of the epipolar line

\[ e_2 \sim K_2 R_2 (t_1 - t_2) \]

\[ q_2 \sim K_2 R_2 (\lambda_1 R_1^T K_1^{-1} q_1 + (t_1 - t_2)) \]
Derivation of the epipolar line

- The epipolar line $l_2$ can be obtained from the cross-product of $e_2$ and $q_2$.
- Note that $Mx \times My \sim M^{-T}(x \times y)$.
- Thus we have:

$$l_2 \sim e_2 \times q_2$$

$$\sim K_2R_2(t_1 - t_2) \times K_2R_2(\lambda_1R_1^TK_1^{-1}q_1 + (t_1 - t_2))$$
Derivation of the epipolar line

\[ \mathbf{e}_2 \times \mathbf{q}_2 \]

\[ \sim (K_2 R_2)^{-T} \left( (\mathbf{t}_1 - \mathbf{t}_2) \times (\lambda_1 R_1^T K_1^{-1} \mathbf{q}_1 + (\mathbf{t}_1 - \mathbf{t}_2)) \right) \]

- Since \( \mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c} \) and \( \mathbf{a} \times \mathbf{a} = \mathbf{0} \), we have:

\[ \mathbf{l}_2 \sim (K_2 R_2)^{-T} \left( (\mathbf{t}_1 - \mathbf{t}_2) \times \lambda_1 R_1^T K_1^{-1} \mathbf{q}_1 \right) \]

We can remove \( \lambda_1 \) since the relation is up to a scale:

\[ \mathbf{l}_2 \sim (K_2 R_2)^{-T} \left( (\mathbf{t}_1 - \mathbf{t}_2) \times R_1^T K_1^{-1} \mathbf{q}_1 \right) \]
Derivation of the epipolar line

\[
Q^m \text{ is epipolar plane}
\]

\[
\mathbf{l}_2 \sim (K_2R_2)^{-T}((t_1 - t_2) \times R_1^TK_1^{-1}q_1)
\]

- Skew-symmetric matrix of any $3 \times 1$ vector $\mathbf{a}$ is given below:

\[
[\mathbf{a}]_\times = \begin{pmatrix}
0 & -a_3 & a_2 \\
-3 & 0 & -a_1 \\
a_2 & a_1 & 0
\end{pmatrix}
\]
Derivation of the epipolar line

\[ l_2 \sim (K_2R_2)^{-T}((t_1 - t_2) \times R_1^T K_1^{-1} q_1) \]

We know that the cross-product of two 3 × 1 vectors \( a \) and \( b \) can be written as follows:

\[ a \times b = [a] \times b \]

\[ l_2 \sim (K_2R_2)^{-T}([t_1 - t_2] \times R_1^T K_1^{-1} q_1) \]
Derivation of the epipolar line

Here we can see the transformation of a point $q_1$ in the left image to a line $l_2$ in the right image using a $3 \times 3$ matrix $(K_2 R_2)^{-T}([t_1 - t_2] \times R_1^T K_1^{-1} q_1)$.

$$l_2 \sim (K_2 R_2)^{-T}([t_1 - t_2] \times R_1^T K_1^{-1} q_1)$$

$$l_2 \sim (K_2 R_2)^{-T}[t_1 - t_2] \times (R_1^T K_1^{-1}) q_1$$
Presentation Outline

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2. Epipolar constraint

3. Fundamental Matrix
The 3×3 matrix is the celebrated fundamental matrix:
\[ F_{12} = (K_2 R_2)^{-T} [t_1 - t_2] \times (R_1^T K_1^{-1}) \]
This matrix encodes the epipolar geometry.
We know that \( \mathbf{q}_2^T \mathbf{l}_2 = 0 \). Thus we have the following:
\[ \mathbf{q}_2^T F_{12} \mathbf{q}_1 = 0 \]
We can have the following equation based on the epipolar line $l_1$

$$\mathbf{q}_1^T \mathbf{F}_{21} \mathbf{q}_2 = 0$$

- For simplicity we will only consider the following equation:

$$\mathbf{q}_2^T \mathbf{F} \mathbf{q}_1 = 0$$

- This constraint is the so-called **epipolar constraint**.
Computation of Fundamental matrix

- Calibration matrices:

\[ K_1 = K_2 = \begin{pmatrix} 200 & 0 & 320 \\ 0 & 200 & 240 \\ 0 & 0 & 1 \end{pmatrix} \]

- Rotation matrices: \( R_1 = R_2 = I \).

- Translation matrices: \( t_1 = \mathbf{0} \), \( t_2 = (100, 0, 0)^T \).

- Correspondences: \( q_1 = (520, 440, 1)^T \), \( q_2 = (500, 440, 1)^T \)

- Compute the fundamental matrix \( F \) and show that \( q_2^T F q_1 = 0 \).

- Find the two epipoles and epipolar lines.
Computation of the fundamental matrix

- Epipolar constraint: $\mathbf{q}_2^T \mathbf{F} \mathbf{q}_1 = 0$
- Using $n$ point correspondences we can rewrite the above equation of the following form:

$$A \mathbf{f} = 0$$

Here $A$ is a $n \times 9$ matrix consisting of only the coordinates of the point correspondences that are known. The $9 \times 1$ vector $\mathbf{f}$ consists of 9 unknowns from the $3 \times 3$ fundamental matrix $\mathbf{F}$. The epipolar plane is shown in the diagram.
Computation of the fundamental matrix

- Using $n$ point correspondences, we can have the following equation:

$$A_{n \times 9} \mathbf{f} = 0$$

$$A = \begin{pmatrix}
q_{1x} q_{2x} & q_{1y} q_{2x} & q_{2x} & q_{1x} q_{2y} & q_{1y} q_{2y} & q_{2y} & q_{2x} & q_{2y} & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{pmatrix}$$

$$\mathbf{f} = ( f_{11} \ f_{12} \ f_{13} \ f_{21} \ f_{22} \ f_{23} \ f_{31} \ f_{32} \ f_{33} )^T$$

- Show the $n \times 9$ matrix using the point correspondences $
\{ \mathbf{q}_1, \mathbf{q}_2 \} = \{(u_{1i}, v_{1i}), (u_{2i}, v_{2i})\}, \ i = \{1 \cdots n\}.$
To find the solution of the equation $Af = 0$, we first compute SVD of $A$, i.e., $[U, S, V] = \text{SVD}(A)$ and then the solution of $f$ is given by the last column of $V$.

The rank of $A$ should be 8 if we use 8 point correspondences.
Some presentation slides are adapted from the following materials:

- Peter Sturm, Some lecture notes on geometric computer vision (available online).