LES of Turbulent Flows: Lecture 2
(ME EN 7960-003)

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Turbulent Flow Properties

- Review from Previous

Properties of Turbulent Flows:

1. **Unsteadiness**:
   
   ![Graph showing unsteady flow](image1)
   
   - $u = f(x,t)$
   
   - Time

2. **3D**:
   
   ![Graph showing 3D flow](image2)
   
   - Contains random-like variability in space
   
   - $x_i$ (all 3 directions)

3. **High vorticity**:
   
   - Vortex stretching mechanism to increase the intensity of turbulence
   
   - (we can measure the intensity of turbulence with the turbulence intensity $\Rightarrow \frac{\sigma_u}{\langle u \rangle}$)

   - **Vorticity**: $\omega = \nabla \times \vec{u}$ or $\omega_k = \epsilon_{ijk} \frac{\partial}{\partial x_i} u_j \hat{e}_k$
Properties of Turbulent Flows:

4. **Mixing effect:**
   Turbulence mixes quantities with the result that gradients are reduced (e.g. pollutants, chemicals, velocity components, etc.). This lowers the concentration of harmful scalars but increases drag.

5. **A continuous spectrum (range) of scales:**

   ![Diagram of eddy scales and energy cascade]

   - **Integral Scale**
   - **Kolmogorov Scale**

   (Richardson, 1922)
Turbulence Scales

• The largest scale is referred to as the Integral scale ($l_o$). It is on the order of the autocorrelation length.

• In a boundary layer, the integral scale is comparable to the boundary layer height.

$\eta$ ($\sim 1$ mm in ABL)  
Kolmogorov micro scale  
(viscous length scale)

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Energy production  
(due to shear)

Energy dissipation  
(due to viscosity)

Energy cascade (transfer) large=>small  
continuous range of scale
Kolmogorov’s Similarity hypothesis (1941)

**Kolmogorov’s 1st Hypothesis:**
- smallest scales receive energy at a rate proportional to the dissipation of energy rate.
- motion of the very smallest scales in a flow depend only on:
  - rate of energy transfer from small scales: \( \epsilon \left( \frac{L^2}{T^3} \right) \)
  - kinematic viscosity: \( \nu \left( \frac{L^2}{T} \right) \)

With this he defined the Kolmogorov scales (dissipation scales):
- length scale: \( \eta = \left( \frac{\nu^3}{\epsilon} \right)^{\frac{1}{4}} \)
- time scale: \( \tau = \left( \frac{\nu}{\epsilon} \right)^{\frac{1}{2}} \)
- velocity scale: \( \nu = (\nu \epsilon)^{\frac{1}{4}} \)

Re based on the Kolmogorov scales => Re=1
Kolmogorov’s Similarity hypothesis (1941)

From our scales we can also form the ratios of the largest to smallest scales in the flow (using $\ell_o$, $U_o$, $t_o$).

Note: dissipation at large scales $\Rightarrow \epsilon \sim \frac{U_o^3}{\ell_o}$

- **length scale:**
  \[ \eta = \left( \frac{\nu^3}{\epsilon} \right)^{\frac{1}{4}} \sim \left( \frac{\nu^3 \ell_o}{U_o^3} \right)^{\frac{1}{4}} \Rightarrow \frac{\eta}{\ell_o^{1/4}} \sim \frac{\nu^{3/4}}{U_o^{3/4}} \Rightarrow \frac{\eta}{\ell_o} \sim \frac{\nu^{3/4}}{U_o^{3/4} \ell_o^{3/4}} \sim Re^{-3/4} \]

- **velocity scale:**
  \[ v = (\nu \epsilon)^{\frac{1}{4}} \sim \left( \frac{\nu U_o^3}{\ell_o} \right)^{\frac{1}{4}} \Rightarrow \frac{v}{U_o^{3/4}} \sim \frac{\nu^{1/4}}{\ell_o^{1/4}} \Rightarrow \frac{v}{U_o} \sim Re^{-1/4} \]

- **time scale:**
  \[ \tau = \frac{\eta}{v} \Rightarrow \frac{\tau}{t_o} \sim Re^{-1/2} \]

For very high-Re flows (e.g., Atmosphere) we have a range of scales that is small compared to $\ell_o$ but large compared to $\eta$. As $Re$ goes up, $\eta / \ell_o$ goes down and we have a larger separation between large and small scales.
Kolmogorov’s Similarity hypothesis (1941)

Kolmogorov’s 2\textsuperscript{nd} Hypothesis:

In Turbulent flow, a range of scales exists at very high Re where statistics of motion in a range $\ell$ (for $\ell_o >> \ell >> \eta$) have a universal form that is determined only by $\epsilon$ (dissipation) and independent of $\nu$ (kinematic viscosity).

- Kolmogorov formed his hypothesis and examined it by looking at the pdf of velocity increments $\Delta u$.

The moments of this pdf are the structure functions of different order (e.g., 2\textsuperscript{nd}, 3\textsuperscript{rd}, 4\textsuperscript{th}, etc.)

• What are structure functions ???
Important single point stats for joint variables

• **covariance:**

\[
cov(U_1, U_2) = \langle u_1 u_2 \rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (V_1 - \langle U_1 \rangle)(V_2 - \langle U_2 \rangle) f_{12}(V_1, V_2) dV_2 dV_1
\]

• Or for discrete data

\[
cov(U_1, U_2) = \langle u_1 u_2 \rangle = \frac{1}{N-1} \sum_{j=1}^{N} (V_{1j} - \langle U_1 \rangle)(V_{2j} - \langle U_2 \rangle)
\]

• We can also define the correlation coefficient (non dimensional)

\[
\rho_{12} = \frac{\langle u_1 u_2 \rangle}{\left[ \langle u_1^2 \rangle \langle u_2^2 \rangle \right]^{1/2}}
\]

• Note that \(-1 \leq \rho_{12} \leq 1\) and negative value mean the variables are anti-correlated with positive values indicating a correlation

• **Practically speaking,** we find the PDF of a time (or space) series by:
  1. Create a histogram of the series (group values into bins)
  2. Normalize the bin weights by the total # of points
Two-point statistical measures

- **autocovariance**: measures how a variable changes (or the correlation) with different lags
  
  \[ R(s) \equiv \langle u(t)u(t + s) \rangle \]

- or the autocorrelation function
  
  \[ \rho(s) \equiv \frac{\langle u(t)u(t + s) \rangle}{\langle u(t)^2 \rangle} \]
  
  - These are very similar to the covariance and correlation coefficient
  - The difference is that we are now looking at the linear correlation of a signal with itself but at two different times (or spatial points), i.e. we lag the series.

- Discrete form of autocorrelation:
  
  \[ \rho(s_j) = \frac{\sum_{k=0}^{N-j-1} (u_ku_{k+j})}{\sum_{k=0}^{N-1} (u_k^2)} \]

- We could also look at the cross correlations in the same manner (between two different variables with a lag).

- Note that: \( \rho(0) = 1 \) and \( |\rho(s)| \leq 1 \)
Two-point statistical measures

- In turbulent flows, we expect the correlation to diminish with increasing time (or distance) between points:

- We can use this to define an Integral time scale (or space). It is defined as the time lag where the integral $\int \rho(s) ds$ converges and can be used to define the largest scales of motion (statistically).

- Another important 2 point statistic is the **structure function**:

$$D_n(r) \equiv \left\langle \left[U_1(x + r, t) - U_1(x, t)\right]^n \right\rangle$$

This gives us the average difference between two points separated by a distance $r$ raised to a power $n$. In some sense it is a measure of the moments of the velocity increment PDF. Note the difference between this and the **autocorrelation** which is **statistical linear correlation** (ie multiplication) of the two points.
Alternatively, we can also look at turbulence in wave (frequency) space:

**Fourier Transforms** are a common tool in fluid dynamics (see Pope, Appendix D-G, Stull handouts online)

Some uses:

- Analysis of turbulent flow
- Numerical simulations of N-S equations
- Analysis of numerical schemes (modified wavenumbers)

consider a periodic function \( f(x) \) (could also be \( f(t) \)) on a domain of length \( 2\pi \)

- The Fourier representation of this function (or a general signal) is:

\[
f(x) = \sum_{k=-\infty}^{k=\infty} \hat{f}_k e^{ikx}
\]

- where \( k \) is the wavenumber (frequency if \( f(t) \))
- \( \hat{f}_k \) are the Fourier coefficients which in general are complex
Fourier Transforms

- Fourier Transform example (from Stull, 88 see example: FourierTransDemo.m)

\[ \text{Real cosine component} \]

\[ \text{Real sine component} \]

\[ \text{Sum of waves} \]
Fourier Transform Applications

Energy Spectrum: (power spectrum, energy spectral density)

• If we look at specific $k$ values from we can define:

$$ E(k) = N \left| \hat{f}_k \right|^2 $$

where $E(k)$ is the energy spectral density

• Typically (when written as) $E(k)$ we mean the contribution to the turbulent kinetic energy (tke) = $\frac{1}{2}(u^2+v^2+w^2)$ and we would say that $E(k)$ is the contribution to tke for motions of the scale (or size) $k$. For a single velocity component in one direction we would write $E_{ij}(k_j)$.

• See supplement for more on Fourier Transforms

The square of the Fourier coefficients is the contribution to the variance by fluctuations of scale $k$ (wavenumber or equivalently frequency)

Example energy spectrum
Kolmogorov’s Similarity Hypothesis (1941)

• Another way to look at this (equivalent to structure functions) is to examine what it means for $E(k)$ where $E(k) \, dk = \text{t.k.e. contained between } k \text{ and } k + dk$

• What are the implications of Kolmogorov’s hypothesis for $E(k)$? $K41 \Rightarrow E(k) = f(k, \varepsilon)$

By dimensional analysis we can find that: $E(k) = c_k \varepsilon^{2/3} k^{-5/3}$

• This expression is valid for the range of length scales $\ell$ where $\ell_o \gg \ell \gg \eta$ and is usually called the inertial subrange of turbulence.

• graphically: