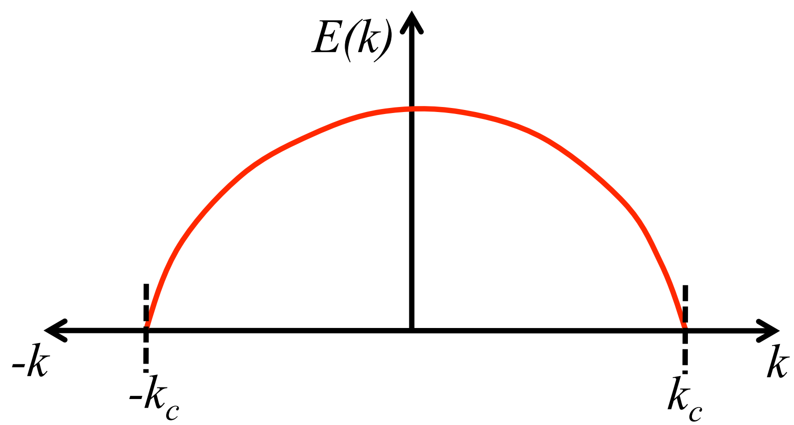
LES of Turbulent Flows: Lecture 5 (ME EN 7960-008)

Prof. Rob Stoll
Department of Mechanical Engineering
University of Utah

Spring 2011

Sampling Theorem

Band-Limited function: a function where $\hat{f}_k = 0 \text{ for } |k| > k_c$



Theorem: If $f(x)$ is band limited, i.e., $\hat{f}_k = 0 \text{ for } |k| > k_c$, then $f(x)$ is completely represented by its values on a discrete grid, $x_n = {n\pi}/{k_c}$ where n is an integer (-$\infty < n < \infty$) and k_c is called the Nyquist frequency.

Implication:

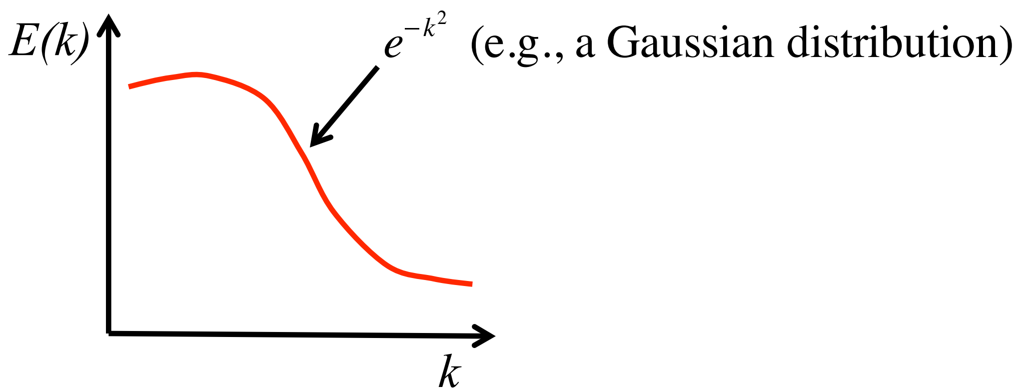
- If we have $x_j = j\,{\pi}/{k_c} = jh \Rightarrow h = {\pi}/{k_c}$ with a domain of 2π: $h = {2\pi}/{N} = {\pi}/{k_c} \Rightarrow k_c = {N}/{2}$

➔ If the number of points is ≥ 2kc then the discrete **Fourier Transform=exact solution**

e.g., for $f(x)=cos(6x)$ we need $N \geq 12$ points to represent the function exactly

Sampling Theorem

- What if f(x) is not band-limited?



- or $f(x)$ is band limited but sampled at a rate < k_c, for example $f(x)=cos(6x)$ with 8 points.

- **Result**: **Aliasing** → contamination of resolved energy by energy outside of the resolved scales.

Aliasing

- Consider: $e^{ik_1 x_j}$ and e^{ik2x_j} and let $k_1 = k_2 + 2mk_c$

where $k_c =$ Nyquist frequency, $m = \pm$ any integer value and $x_j = j\,\pi/k_c$

$$e^{ik_1 x_j} = e^{i(k_2 + 2mk_c)x_j}$$

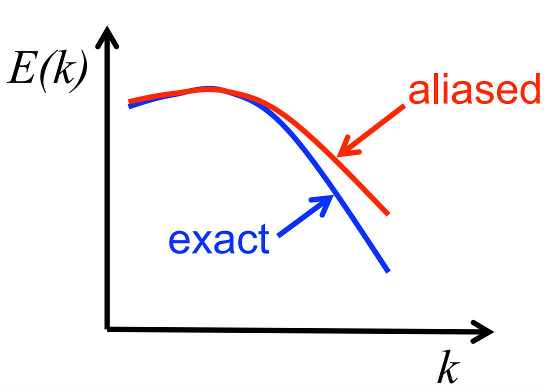
$$= e^{ik_2 x_j} e^{2mk_c x_j} = e^{ik_2 x_j} e^{2mk_c j \pi/k_c}$$

$$= e^{ik_2 x_j} \underbrace{e^{i2\pi mj}}_{=1}$$

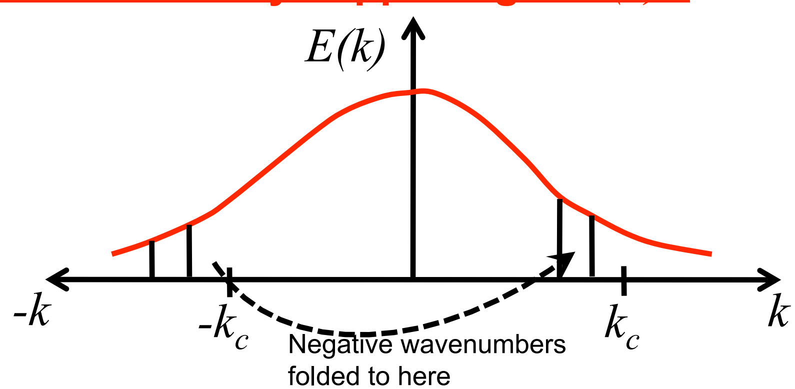
=1 (integer function of 2π)

$e^{ik_1 x_j} = e^{ik_2 x_j} \Rightarrow$ result is that we can't tell the difference between k_2 and $k_1 = k_2 + 2mk_c$ on a discrete grid. k_1 is aliased onto k_2

What does this mean for spectra?



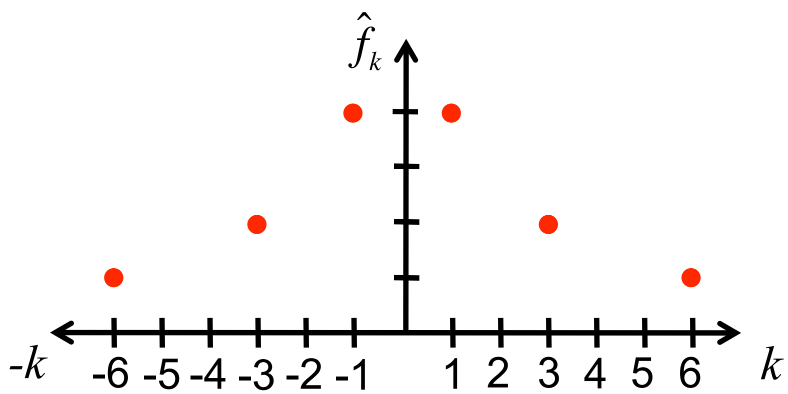
What is actually happening to *E(k)*?



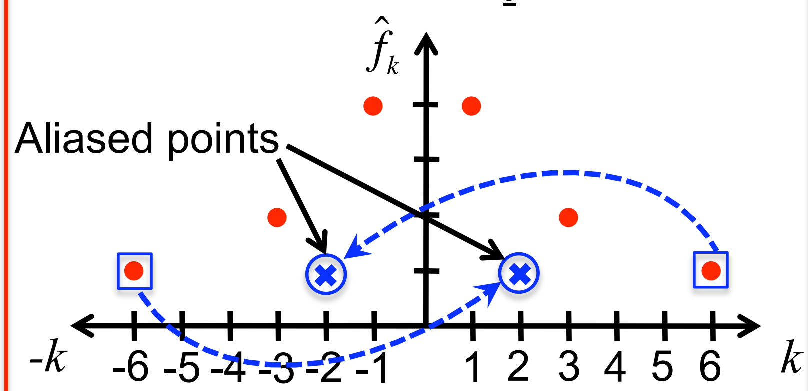
Aliasing Example

Consider a function: $\text{e.g.,} f(x) = \cos(x) + \tfrac{1}{2}\cos(3x) + \tfrac{1}{4}\cos(6x)$

Fourier coefficients (all real since only cosine)



Consider N=8 ➔ $k_c = N/2 = 4$



$$\text{Aliasing } \Rightarrow k_1 = k_2 + 2mk_c = k_2 + 8m \Rightarrow -6 \text{ gets aliased to } 2$$

$$\text{and if } m = -1 \Rightarrow k_1 = k_2 - 8 \Rightarrow 6 \text{ gets aliased to -2}$$

Aliasing ↓ if N ↑

For more on Fourier Transforms see Pope Ch. 6, online handout from Stull or Press et al., Ch 12-13.

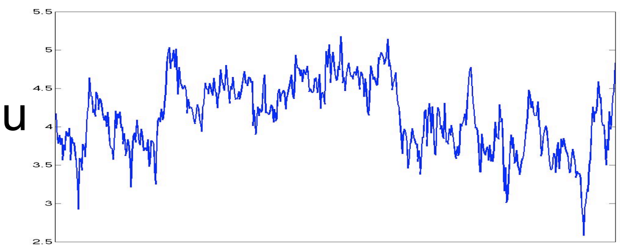
Turbulent Flow Properties

- Why study turbulence? Most real flows in engineering applications are turbulent.

Properties of Turbulent Flows:

1. <u>Unsteadiness:</u>

u

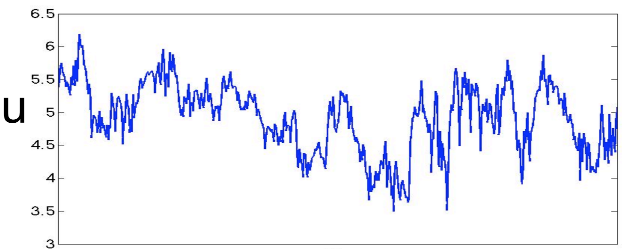


u=f(**x**,t)

time

2. <u>3D:</u>

u



contains random-like variability in space

x_i (all 3 directions)

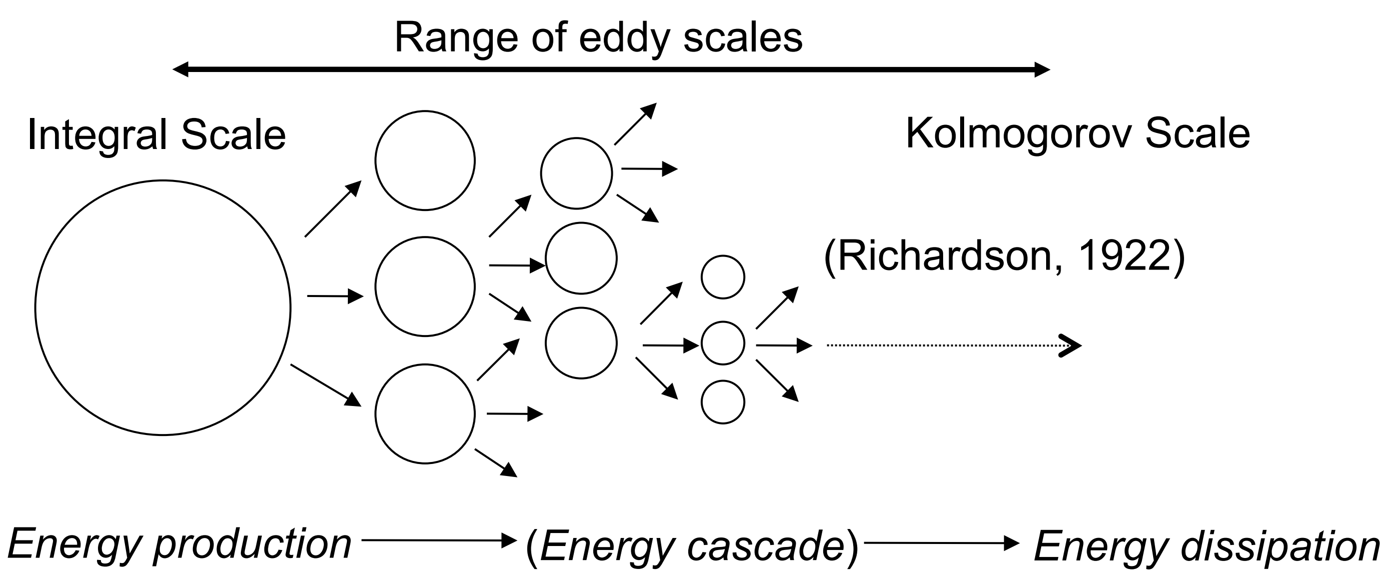
3. <u>High vorticity:</u>
   Vortex stretching → mechanism to increase the intensity of turbulence
   (we can measure the intensity of turbulence with the turbulence intensity => $\frac{\sigma_u}{\langle u \rangle}$) Vorticity: $\omega = \vec{\nabla} \times \vec{u}$ or $\epsilon_{ijk}\frac{\partial}{\partial x_i}u_j\hat{e}_k$

Turbulent Flow Properties (cont.)

Properties of Turbulent Flows:

4. Mixing effect:
   Turbulence mixes quantities with the result that gradients are reduced (e.g. pollutants, chemicals, velocity components, etc.). This lowers the concentration of harmful scalars but increases drag.

5. A continuous spectrum (range) of scales:



Turbulence Scales

- The largest scale is referred to as the Integral scale (l_o). It is on the order of the autocorrelation length.

- In a boundary layer, the integral scale is comparable to the boundary layer height.