LES Lecture 18

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→ 1st point out

Homogeneous ~~Isotropic~~ Turbulence:

Important test case for turbulent flow modeling.

- used with DNS to study basics of the energy cascade.
- used with LES to test model performance (a posteriori) and to study model physics (a priori)

– A few basic cases are usually used in studies: homogeneous shear flow, homogeneous rotating turbulence, forced isotropic turbulence, $\boxed{\text{decaying isotropic turbulence}}$ (see Pope ch. 5 for some of these and other examples)

Basic Properties:

homogeneous – statistics do not depend on position

Isotropic – statistics do not depend on orientation.

⇒ Isotropic ⇒ homogeneous.

- In a laboratory the closest analogy to Isotropic turbulence is grid turbulence:

- ~~1 lot grid~~ • if we define $u_i = \langle U_i \rangle + u_i'$

$\langle u_i \rangle$ is also uniform and can be assumed $= 0$ without loss of generality (recall Galilean property)

$\Rightarrow u_i = u_i'$

- recall we can define a Reynolds stress $\langle u_i' u_j' \rangle = R_{ij}$
- Homogeneous implies $R_{ij} = R_{ij}(t)$ and for isotropic

$\boxed{R_{ij} \; i \neq j = 0}$ ⇒ no off diagonal components! (would imply shear)

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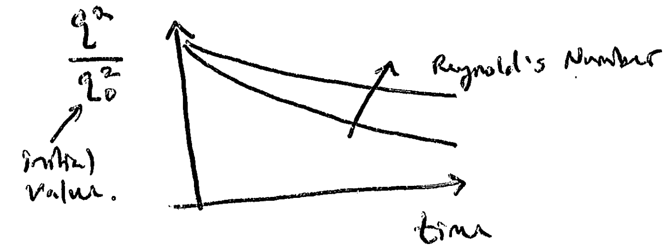
kinetic energy

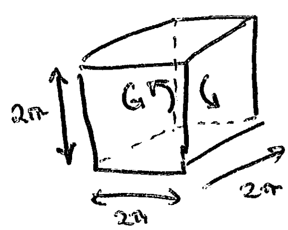
$$q^2 = \text{trace of } R_{ij} = R_{ii} = \langle u'^2 \rangle + \langle v'^2 \rangle + \langle w'^2 \rangle = 2k$$

(kinetic energy)

typical plot of this:



- For one of the projects, you will apply LES SGS models in an a posteriori study of this case.
- Domain is a cube of 2π on each side



- periodic (homogeneous!) in all 3 directions

- our goal is to simulate the filtered N-S equations.

$$\frac{\partial \hat{u}_i}{\partial t} + \frac{\partial}{\partial x_j}\hat{u}_i\hat{u}_j = -\frac{\partial P}{\partial x_i} + \nu\frac{\partial^2 \hat{u}_i}{\partial x_j^2} - \frac{\partial \tau_{ij}}{\partial x_j}$$

- you will be provided with a basic code that solves the 3D (unfiltered) N-S equations $\Rightarrow$

$$\frac{\partial u_i}{\partial t} + \frac{\partial}{\partial x_j}u_i u_j = -\frac{\partial p}{\partial x_i} + \nu\frac{\partial^2 u_i}{\partial x_j}$$

on our $2\pi^3$ periodic grid using spectral methods.

- you will need to add the $\frac{\partial \tau_{ij}}{\partial x_j}$ term to the code!

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- 1^{st} a few notes about spectral methods (how they work)
(brief since I will provide)

Taking derivatives:

1^{st} deriv: if the FT of $f_j = \hat{f}_k$

then what is the FT of $\left.\frac{\partial f_j}{\partial x}\right|_j =$

- look at discrete representation $\Rightarrow f_j = \sum_{k=-\frac{N}{2}}^{\frac{N}{2}-1} \hat{f}_k e^{ikx_j}$
- take $\frac{\partial}{\partial x} \Rightarrow \left.\frac{\partial f}{\partial x}\right|_j = \sum_{k=-N/2}^{\frac{N}{2}-1} ik\,\hat{f}_k e^{ikx_j}$

$$\Rightarrow \widehat{\frac{\partial f}{\partial x}} = ik\underbrace{\hat{f}_k}_{\text{fourier coefficients.}}$$

$\Rightarrow$ in fourier space we can compute the derivative by multiplying by the wavenumber and i

2^{nd} deriv: by the same reasoning we can show that

$$\widehat{\frac{\partial^2 f}{\partial x}} = \underbrace{-k^2}_{\text{squared wavenumber.}} \hat{f}_k \qquad i^2 = -1$$

Fourier Transform of Products

- we also have terms that look like $u_i u_j$ products!
- we can show (see Canuto et al) that for $F = fg \Rightarrow \hat{F}_k = \sum_{k'} \hat{f}_{k'}\, \hat{g}_{k-k'} \approx O(N^2)$ for each wavenumber!
- result $\Rightarrow$ something simple in real space is very expensive in wavespace

$\Rightarrow$ pseudo spectral methods: use a mix of operations in real space and wavespace products $\Rightarrow$ real space!

- a problem with this ... Alaising !!
- solution => use the 3/2 rule we developed earlier when we talked about sampling theorem.
  - take variables to $3/2 N$ grid using (in real space) fourier interpolation
  - compute products (or division)
  - go back to wave space and truncate wavenumbers

[the code you will be given has this as a sub function dealias]

(*) Time advancement : Two options

- Galerkin approach : use discrete orthogonality and integrate in time (example in a second) in fourier space (solution satisfied at every wavenumber)
- Colocation approach: require equation is discretely satisfied at every point (still use FT for derivs).
- The code provided uses a Galerkin approach with a 4^{th} order Runge-Kutta scheme

e.g.

$$y_{n+1} = y_n + \frac{1}{6} k_1 + \frac{1}{3}(k_2 + k_3) + \frac{1}{6} k_4$$

$$k_1 = h f(y_n, t_n)$$

$$k_2 = h f\left(y_n + \frac{k_1}{2}, t_n + \frac{h}{2}\right)$$

$$k_3 = h f\left(y_n + \frac{k_2}{2}, t_n + \frac{h}{2}\right)$$

$$k_4 = h f(y_n + k_3, t_n + h)$$

(see Ferziger and Peric for details)

- Back to N-S equations:
  - The DNS code I will provide solves:

$$\frac{\partial u_i}{\partial t} + \frac{\partial}{\partial x_j} u_i u_j = -\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j^2} \quad \text{and} \quad \frac{\partial u_i}{\partial x_i} = 0$$

its periodic in 3D (isotropic turbulence must be) spectral methods

$$\Rightarrow \quad u_i(x,y,z,t) = \sum_{k_1=-\frac{N_1}{2}}^{\frac{N_1}{2}-1} \sum_{k_2=-\frac{N_2}{2}}^{\frac{N_2}{2}-1} \sum_{k_3=-\frac{N_3}{2}}^{\frac{N_3}{2}-1} \hat{u}_i(\vec{k},t)\, e^{i\vec{k}\cdot\vec{x}}$$

and

$$p(\vec{x},t) = \sum_{k_1}\sum_{k_2}\sum_{k_3} \hat{p}(\vec{x},t)\, e^{i\vec{k}\cdot\vec{x}}$$

- if we substitute these into momentum equation (or you can think of this as taking the Fourier Trans of N-S)

$$\frac{\partial}{\partial t}\sum_{\vec{k}} \hat{u}_i e^{i\vec{k}\cdot\vec{x}} + \frac{\partial}{\partial x_j}\sum_{\vec{k}} \widehat{u_i u_j}\, e^{i\vec{k}\cdot\vec{x}} = -\frac{\partial}{\partial x_i}\sum_{\vec{k}} \hat{p}\, e^{i\vec{k}\cdot\vec{x}} - \nu \frac{\partial^2}{\partial x_j^2}\sum_{\vec{k}} \hat{u}_i e^{i\vec{k}\cdot\vec{x}}$$

- now we do two things 1) use discrete orthogonality of FT (ie fourier modes are orthogonal and independent $\Rightarrow$ this equation must be satisfied at every wavenumber) 2) use our differentiation in fourier space

$$\Rightarrow \quad \frac{\partial \hat{u}_i}{\partial t} + i k_j \widehat{u_i u_j} = -i k_i \hat{p} - \nu \overbrace{k_j k_j}^{k^2} \hat{u}_i$$

and for the continuity equation in Fourier Space $i k_i \hat{u}_i = 0$ or $k_i u_i = 0$

$\Rightarrow$ contract momentum with $i k_i$ (ie take divergence)

$$\frac{\partial}{\partial t} \underbrace{\cancel{i k_i u_i}}_{\text{cont.}} + \overbrace{i^2}^{-1} k_i k_j \widehat{u_i u_j} = -\overbrace{i^2}^{-1} \overbrace{k_i k_i}^{k^2} \hat{p} - \nu k^2 \underbrace{\cancel{i k_i u_i}}_{\text{cont.}}$$

$$\Rightarrow \quad \boxed{\hat{p} = -\frac{k_i k_j}{k^2} \widehat{u_i u_j}}$$

Algebraic Pressure! in fourier Space

- substituting our pressure back into mom

$$\frac{\partial \hat{u}_i}{\partial t} = -ik_j \widehat{u_i u_j} + ik_i \underbrace{\frac{k_\ell k_m}{k^2} \widehat{u_\ell u_m}}_{\substack{\text{note all} \\ \text{terms in} \\ \text{all 3 equations.}}} - \nu k^2 \hat{u}_i$$

- So what steps does the code follow??

1) specify initial conditions (note BCs set by default)

2) compute $\widehat{u_i u_j}$ using pseudo spectral

dealias 1 { - compute $\hat{u}_i$ on N intervals
- expand to $\frac{3N}{2}$ by zero padding
- Inverse FFT $\hat{u}_i$ on $3/2 N$ }

- compute $u_i u_j$ on $3/2 N$ (real space)

dealias 2 { - FT $u_i u_j$ on $3/2 N$ grid
- truncate $\widehat{u_i u_j}$ to N }

3) assemble RHS

4) integrate using RK4

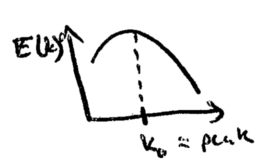
- only Thing left: what are the initial conditions??

- Require: Real, isotropic, Divergence free
(details see Rogallo 1981)

1) Choose a idealized initial spectrum

$$E(k) = 16\sqrt{\frac{2}{\pi}} \frac{u_0^2}{k_0} \left(\frac{k}{k_0}\right)^4 e^{-2\left(\frac{k}{k_0}\right)^2}$$

which has the form:



and for this spectrum

$$\int_0^\infty E(k)\, dk = 3/2\, u_0^2$$

2) choose u_0 k_0

3) choose ν such that $R_\lambda = \frac{u_0 \lambda}{\nu}$ = specified value

$\lambda \equiv$ taylor microscale

$$\lambda_\alpha = \sqrt{\frac{\overline{u_\alpha^2}}{\overline{\left(\frac{\partial u_\alpha}{\partial x_\alpha}\right)^2}}}$$

(any direction)

- for our chosen spectrum $\lambda = 2/k_0$
- for a res of $\sim 32^3$ DNS at R_λ of about 20-30 works $\Rightarrow$ this will specify ν

→ things you must compute (at least)

- spectra at different times
- kinetic energy (with time)
- velocity derivative skewness

$$S_k = \overline{\left(\frac{\partial u}{\partial x}\right)^3} \Big/ \left[\overline{\left(\frac{\partial u}{\partial x}\right)^2}\right]^{3/2}$$