LES LECTURE 23 (page 1)

Reminder papers discuss presentation schedule.

our exam is Wed 4th 8-10 am
↳ 120 minutes

14 people to present => 2 class periods.

- Paper format

- 1st go over BC evaluations...

- PDF methods (Rest of class will be this)
  (~~and~~ we will mostly follow Pope Ch. 12)

Review Definitions for PDFs

PDF of vel. at pos. x and t in sample space V for velocity U

- $$\int f(\vec{V}; \vec{x}, t)\, d\vec{V} = 1$$

- mean (or expected) value is defined by

$$\langle Q(U(\vec{x},t))\rangle = \int Q(\vec{V})\, f(\vec{V}; x, t)\, d\vec{V}$$

=> mean velocity is simply.

$$\langle U(\vec{x},t)\rangle = \int \vec{V} f(\vec{V}; x, t)\, d\vec{V}$$

and

$$\langle u_i u_j\rangle = \int (V_i - \langle U_i\rangle)(V_j - \langle u_j\rangle)\, f(V; x, t)\, dV$$

- in the PDF transport equation unknowns appear as conditional means.

$\phi(x,t) \equiv$ random field

$f_{u\phi}(V, \varphi; \vec{x}, t) \equiv$ (one point, one time) joint PDF of U and ϕ (φ → sample space of ϕ)

PDF of $\phi(x,t)$ conditional on $U(\vec{x},t) = V$ is

$$f_{\phi|U}(\varphi|\vec{V}, \vec{x}, t) = \frac{f_{U\phi}(\vec{V}, \varphi; \vec{x}, t)}{f(\vec{V}; \vec{x}, t)} \qquad (*)$$

and the conditional mean value is:

$$\langle \phi(\vec{x},t) | U(\vec{x},t) = V \rangle = \int \varphi \, f_{\phi|U}(\varphi|\vec{V}, \vec{x}, t)\, d\varphi$$

many times written as

$$\langle \phi | \vec{V} \rangle$$

~~From~~ this conditional mean ~~~~ can be associated with ~~to~~ the actual mean by

$$\langle \phi(x,t) \rangle = \iint \varphi \, f_{U\phi}(\vec{V}, \varphi; x, t)\, d\varphi\, d\vec{V}$$

use $(*)$

$$= \int \langle \phi(\vec{x},t) | U(\vec{x},t) = V \rangle f(\vec{V}; \vec{x}, t)\, d\vec{V}$$

$$= \int \langle \phi | \vec{V} \rangle f \, d\vec{V} \quad \text{(in shorthand)}$$

PDF Transport equation (Pope appendix H)

- derivation relies on the idea of a fine grained PDF
- $$f'(\vec{V};\vec{x},t) = \delta(\vec{U}(\vec{x},t)-\vec{V}) = \prod_{i=1}^{3} \delta(U_i(\vec{x},t)-V_i)$$

at ~~every~~ every point and time in the flow field f' is a delta function at $U(x,t) = V$
↳ appendix C

- Two properties of fine grained PDF

$$\langle f'(\vec{V};\vec{x},t)\rangle = f(\vec{V};\vec{x},t) \quad \text{and}$$

$$\langle \phi(\vec{x},t)\, f'(\vec{V};\vec{x},t)\rangle = \langle \phi(\vec{x},t) \,|\, U(\vec{x},t) = \vec{V}\rangle f(\vec{V};\vec{x},t)$$

this comes from (do before other relation)

$$\langle f'(\vec{V};\vec{x},t)\rangle = \langle \delta(U(x,t)-\vec{V})\rangle$$
$$= \int \delta(\vec{V}'-\vec{V})\, f(\vec{V}';x,t)\, dV'$$
$$= f(\vec{V}; x,t)$$

shifting property
$$\boxed{\int g(\vec{x})\,\delta(\vec{x}-\vec{y}) = g(y)}$$

obtained in the same way (see Pope)

- derivatives of fine-grained PDFs.

using Pope's example: take $f'_u(v;t) = \delta(u(t)-v)$
scalar process $u(t)$
sample-space variable

even $[f(x) = f(-x)]$
odd $[-f(x) = f(-x)]$

a constant
notation for deriv.

$$\frac{d}{dv}\delta(v-a) = \frac{d}{dv}\delta(a-v) \Rightarrow \delta^{(1)}(v-a) = -\delta^{(1)}(a-v)$$

(derivative is an odd function)

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- differentiating our scalar fine grained PDF

<u>Chain Rule</u> $\left(\frac{dy}{dx} = \frac{dy}{du} * \frac{du}{dx}\right)$

$$\frac{\partial}{\partial t} f_u'(v;t) = \delta^{(1)}(u(t)-v)\frac{du(t)}{dt} = -\delta^{(1)}(v-u(t))\cdot\frac{du(t)}{dt}$$

$$= -\frac{\partial f_u'(v;t)}{\partial v}\frac{du(t)}{dt} = -\frac{\partial}{\partial v}\left(f_u'(v;t)\frac{du(t)}{dt}\right)$$

$u(t)$ is independent of v

- we also need that

$$(**)\quad U_i(\vec{x},t)\frac{\partial}{\partial x_i} f'(\vec{V};\vec{x},t) = \frac{\partial}{\partial x_i}\left[U_i(\vec{x},t) f'(\vec{V};\vec{x},t)\right]$$

$$= \frac{\partial}{\partial x_i}\left[V_i f'(\vec{V};\vec{x},t)\right] = V_i\frac{\partial}{\partial x_i} f'(\vec{V};\vec{x},t)$$

- uses incompress (to bring into deriv)
- shifting property of δ functions
- V_i is independent (can move out)

Derivation of PDF Transport Eqn.

substantial (or total) Derivative of f'

$$* \quad \frac{Df'}{Dt} = \frac{\partial f'}{\partial t} + V_i\frac{\partial f'}{\partial x_i} = -\frac{\partial}{\partial V_i}\left(f'\frac{DU_i}{Dt}\right)$$

we get this using the equivalent of our scalar derivation of f' in time and space (for scalar f')

* The mean of this is

$$\frac{\partial f}{\partial t} + V_i\frac{\partial f}{\partial x_i} = -\frac{\partial}{\partial V_i}\left(f\left\langle\frac{DU_i}{Dt}\middle|\vec{V}\right\rangle\right)$$

(for incompressible flow since ** used)

this is general and has <u>no physics</u> (we haven't said anything about the medium)

we insert physics by substituting in our N-S definition of $\frac{DU_i}{Dt}$ (typical RHS of N-S) with our N-S equations

$$\frac{\partial f}{\partial t} + V_i \frac{\partial f}{\partial x_i} = -\frac{\partial}{\partial V_i}\left(f \left\langle \nu \nabla^2 v_i - \frac{1}{\rho}\frac{\partial p}{\partial x_i} \,\middle|\, \vec{V} \right\rangle \right)$$

if we decompose pressure as

$$\left\langle \frac{\partial p}{\partial x_i} \,\middle|\, \vec{V} \right\rangle = \frac{\partial \langle p \rangle}{\partial x_i} + \left\langle \frac{\partial p'}{\partial x_i} \,\middle|\, \vec{V} \right\rangle$$

we get

$$\frac{\partial f}{\partial t} + V_i \frac{\partial f}{\partial x_i} = \frac{1}{\rho}\frac{\partial \langle p \rangle}{\partial x_i}\frac{\partial f}{\partial V_i} - \frac{\partial}{\partial V_i}\left[f \left\langle \nu \nabla^2 v_i - \frac{1}{\rho}\frac{\partial p'}{\partial x_i} \,\middle|\, \vec{V} \right\rangle \right]$$