Transient Conduction: Semi-Infinite Solids

Reminders...

- Homework #5 due Friday
  - For #5, make it simpler by using $h = 198 \text{ W/m}^2\cdot\text{K}$
  - For #4, temperature is 550°C
  - Help session this afternoon at 4:30 in MEB 2325
  - Returned Monday

- Friday we start working on the project

- Midterm #1 coming up Wed. October 1
  - Covers chapters (1, 2) 3, 4, 5
  - Won’t cover section 5.10 (FDA of transient conduction)

- Midterm Review Session Monday, Sept. 29

- Survey...
Review of 5.1 to 5.6

- Lumped Capacitance Method and Validity
  - Assumes solid is spatially uniform,
  - Resistance to conduction within the solid is small compared to the resistance to transfer between the solid and its surroundings

![Diagram of heat transfer](image-url)

**Figure 5.3**
Effect of Biot number on steady-state temperature distribution in a plane wall with surface convection.
Spatial Effects

- Arises from inadequate solution using lumped capacitance method
  - Temperature gradients are no longer negligible in the medium
- Requires initial and boundary conditions
- Exact solutions involve infinite series
- Approximate solutions use only first term
  - Use Table 5.1 to determine $C_1$ and $\zeta_1$
  - Can use the one-term approximation when $Fo > 0.2$
  - Equations for time, temperature, position, and fraction of total energy transfer for walls, cylinders, and spheres

Example – Book Problem 5.39

The 150-mm-thick wall of a gas-fired furnace is constructed of brick ($k = 1.5$ W/m·K, $\rho = 2600$ kg/m$^3$, $c_p = 1000$ J/kg·K) and is well insulated at its outer surface. The wall is at an initial temperature of 20°C when the burners are fired and the inner surface is exposed to products of combustion for which $T_\infty = 950°C$ and $h = 100$ W/m$^2$·K.

How long does it take for the outer surface of the wall to reach 750°C?

\[
Bi = \frac{hL}{k} = \frac{(100)(0.15)}{1.5} = 10
\]

- Assume $Fo > 0.2$
- Use the for plane wall:
  \[
  \theta_o^* = C_1 \exp\left(-\zeta_1^2 Fo\right)
  \]
- Solve for $t$ via $Fo$
  \[
  Fo = \frac{kt}{\rho c_p L^2} = \frac{\ln\left(\theta_o^* / C_1\right)}{\zeta_1^2}
  \]
Table 5.1 – $\zeta_1$ and $C_1$ vs. Bi

<table>
<thead>
<tr>
<th>$Bi$</th>
<th>Plane Wall</th>
<th>Infinite Cylinder</th>
<th>Sphere</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\zeta_1$ (rad)</td>
<td>$C_1$</td>
<td>$\zeta_1$ (rad)</td>
</tr>
<tr>
<td>0.01</td>
<td>0.0998</td>
<td>1.0017</td>
<td>0.1412</td>
</tr>
<tr>
<td>0.02</td>
<td>0.1410</td>
<td>1.0033</td>
<td>0.1995</td>
</tr>
<tr>
<td>0.03</td>
<td>0.1723</td>
<td>1.0049</td>
<td>0.2440</td>
</tr>
<tr>
<td>0.04</td>
<td>0.1987</td>
<td>1.0066</td>
<td>0.2814</td>
</tr>
<tr>
<td>0.05</td>
<td>0.2218</td>
<td>1.0082</td>
<td>0.3143</td>
</tr>
<tr>
<td>0.06</td>
<td>0.2425</td>
<td>1.0098</td>
<td>0.3438</td>
</tr>
<tr>
<td>0.07</td>
<td>0.2615</td>
<td>1.0114</td>
<td>0.3709</td>
</tr>
<tr>
<td>0.08</td>
<td>0.2791</td>
<td>1.0130</td>
<td>0.3960</td>
</tr>
<tr>
<td>0.09</td>
<td>0.2956</td>
<td>1.0145</td>
<td>0.4195</td>
</tr>
<tr>
<td>0.10</td>
<td>0.3111</td>
<td>1.0161</td>
<td>0.4417</td>
</tr>
<tr>
<td>0.15</td>
<td>0.3779</td>
<td>1.0237</td>
<td>0.5376</td>
</tr>
</tbody>
</table>

$Bi = \frac{Al}{k}$ for the plane wall and $Br/\varepsilon$ for the infinite cylinder and sphere. See Figure 5.6.

Temperature Distribution over Time

Problem 5.39

![Graph of Temperature Distribution over Time](image)
Approximate Solutions for Cylinders and Spheres

- Similar approach as for plane wall
  - NOTE: Use $r_o$ for calculation of Bi and use that Bi to look up values in the table

- Cylinder: 
  \[ \theta^* = \theta_o^* J_0(\zeta_1 r^*) \]
  with centerline $T$: 
  \[ \theta_o^* = C_1 \exp (-\zeta_1^2 Fo) \]
  and total energy transfer: 
  \[ \frac{Q}{Q_o} = 1 - \frac{2\theta_o^*}{\zeta_1 J_1(\zeta_1)} \]

...and for spheres

- Sphere: 
  \[ \theta^* = \theta_o^* \frac{1}{\zeta_1 r^*} \sin (\zeta_1 r^*) \]
  with center $T$: 
  \[ \theta_o^* = C_1 \exp (-\zeta_1^2 Fo) \]
  and total energy transfer:
  \[ \frac{Q}{Q_o} = 1 - \frac{3\theta_o^*}{\zeta_1^3} \left[ \sin (\zeta_1) - \zeta_1 \cos (\zeta_1) \right] \]
5.7 The Semi-Infinite Solid

- An idealized geometry to analytically solve heat transfer problems
- Method for transient conduction problems
- Analytical derivation is found on page 284-286 (6th ed.) or 311-314 (7th ed.)
- Three cases
  - Constant surface temperature
  - Constant surface heat flux
  - Surface convection

**Figure 5.7** Transient temperature distributions in a semi-infinite solid for three surface conditions: constant surface temperature, constant surface heat flux, and surface convection.
Three Cases

Case 1  Constant Surface Temperature: \( T(0, t) = T_s \)

\[
\frac{T(x, t) - T_s}{T_i - T_s} = \text{erf}\left( \frac{x}{2\sqrt{\alpha t}} \right)
\]

\[q_s''(t) = \frac{k(T_s - T_i)}{\sqrt{\pi\alpha t}}\]

Case 2  Constant Surface Heat Flux: \( q_i'' = q_o'' \)

\[
T(x, t) - T_i = \frac{2q_o''(\alpha t/\pi)^{1/2}}{k} \exp\left( \frac{-x^2}{4\alpha t} \right) - \frac{q_o''x}{k} \text{erfc}\left( \frac{x}{2\sqrt{\alpha t}} \right)
\]

Case 3  Surface Convection: \( -k \frac{\partial T}{\partial x} \bigg|_{x=0} = h[T_{\infty} - T(0, t)] \)

\[
\frac{T(x, t) - T_i}{T_{\infty} - T_i} = \text{erfc}\left( \frac{x}{2\sqrt{\alpha t}} \right)
\]

\[
- \left[ \exp\left( \frac{hx}{k} + \frac{h^2\alpha t}{k^2} \right) \right] \text{erfc}\left( \frac{x}{2\sqrt{\alpha t}} + \frac{h\sqrt{\alpha t}}{k} \right)
\]

Gaussian Error Function

- Appendix B.2 (page 961 in 6th ed.)

\[
erf(w) = \frac{2}{\sqrt{\pi}} \int_0^w e^{-v^2} dv
\]

\[\text{erfc}(w) \equiv 1 - \text{erf}(w)
\]
Slabs of Different Temperature

- Temperature slopes are not linear, indicative of transient behavior

Example – Book Problem 5.68

Asphalt pavement may achieve temperatures as high as 50°C on a hot summer day. Assume that such a temperature exists throughout the pavement, when suddenly a rainstorm reduces the surface temperature to 20°C. Calculate the total amount of energy (J/m²) that will be transferred from the asphalt over a 30-minute period in which the surface is maintained at 20°C.